TESTING THE THIN CAPILLARY MODEL

PROBANDO EL MODELO DE CAPILAR DELGADO

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We provide a detailed and accessible introduction to the exact theory of capillary phenomena in a cylindrical tube of finite radius. We demonstrate explicitly that *Jurin's law* always somewhat underestimates the height of liquid rise. The largest deviations occur for small contact angles. In the limiting case of a very large capillary radius, the true column height becomes independent of the radius and approaches a finite limiting value. The mean curvature of a real meniscus is a monotonically increasing function of the capillary radius. We argue that the force balance method is most appropriate for deriving *Jurin's law*, whereas the exact capillary equation requires the use of the pressure balance method or the Helmholtz energy minimization procedure.

Proveemos una accesible y detallada traducción para la teoría exacta de los fenómenos capilares en un tubo cilíndrico de radio finito. Demostramos explícitamente que la *ley de Jurin* siempre subestima la altura que alcanza el líquido. Las mayores desviaciones ocurren para ángulos de contacto pequeños. En el caso límite de un radio capilar muy grande, la altura real de la columna no depende del radio y se acerca a un valor límite finito. La curvatura media de un menisco real es una función monótona decreciente del radio capilar. Discutimos que el método de equilibrio de fuerzas es más apropiado para obtener la *ley de Jurin*, mientras que la ecuación exacta para la capilaridad requiere el uso del método de equilibrio de presiones o la minimización de la energía de Helmholtz.

Keywords: capillary (capilaridad), Jurin's law (ley de Jurin), column height (altura de columna), mean curvature (curvatura media).

I. INTRODUCTION

Capillarity (from the Latin capillaris - "resembling hair") is the process by which liquid flows in a narrow space without the help of or even against any external force such as gravity. The capillary effect occurs due to intermolecular interactions between a liquid and the surface of a solid. If the diameter of the vessel is small enough, the combination of surface tension (which is caused by cohesion between the liquid molecules) and adhesion between the liquid and the walls of the vessel causes the liquid to rise. The first recorded description of capillarity belongs to Leonardo da Vinci. Capillary action underlies the function of wet cloths, sponges, towels, napkins, capillary pens, and lighters.

The simplest analysis of capillary action (Jurin's law) is presented in every introductory physics course. Our experience shows that many students leave these courses with the misconception that Jurin's law is a universal law and can be applied to an arbitrary tube radius. At the same time, some other students repeatedly asked how Jurin's model should be modified for the case of intermediate or even very wide tubes.

A rigorous treatment of the liquids rise within both planar and axisymmetric models is presented in excellent paper of Sai Liu et al [1]. Unfortunately, their numerical calculations are limited to water in glass tubes, reducing the generalizability of the findings. In this paper, we provide a detailed and accessible introduction to the exact theory of capillary phenomena in a cylindrical tube of finite radius. Our approach relies on just two dimensionless parameters, making the theory both practical and broadly applicable. We intend this contribution to be useful to undergraduate university students of physics and chemistry.

II. THE ELEMENTARY CONSIDERATION

Let us recall how Jurin's law can be derived from the force balance method. We consider a thin capillary tube open to the atmosphere with an internal radius r_0, in which there is a liquid column (figure 1).

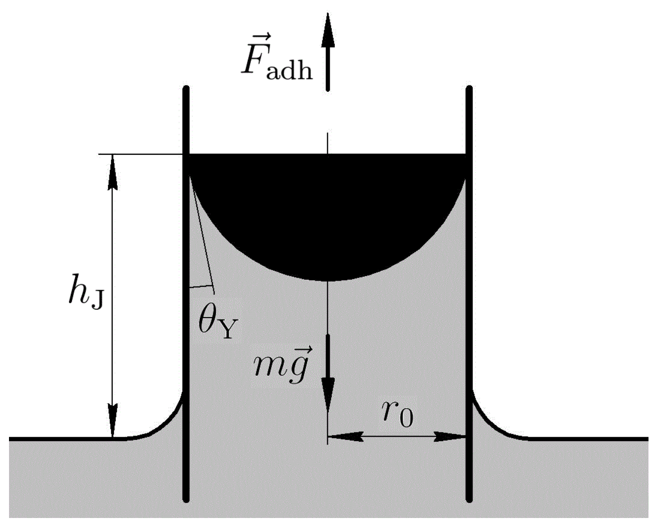

Figure 1. Layout for the derivation of Jurin's law based on force balance.

The total adhesion force acting upward on a three-phase contact line is derived in Ref. [2] and equal to $2\pi\gamma r_0 \cos\theta_{\rm Y}$, where γ is the surface tension; $\theta_{\rm Y}$ ($0 < \theta_{\rm Y} < \pi$) is Young's contact angle of the solid wall.

The contact angles depend on the nature of the liquid, the solid surface, and the surrounding medium (usually air). For example, water wets clean glass well, so the contact angle is close to 0° (almost complete wetting). For perfectly clean glass the angle can be around 0–10°, but in practice (due to contamination) it can reach 20–30°. Mercury does not wet glass, and the contact angle is large, usually around 130–140° (non-wetting). This is due to the high surface energy of mercury and low adhesion to glass.

Under the equilibrium condition, this force should be compensated by the gravity force acting on the liquid. Here is the key Jurin's simplification: we assume that the column height h_J is so large that we can ignore the absence of liquid in the shaded part above the meniscus and take the gravity force equal to $\rho\pi r_0^2 h_\mathrm{J} g$, where ρ is the density of the liquid. Then

$$h_\mathrm{J} = \frac{2\gamma\cos\theta_\mathrm{Y}}{\rho g r_0}. \quad (1)$$

Equation (1) is the mathematical expression of Jurin's law, which states that the maximum height of a liquid in a capillary tube is inversely proportional to the tube's diameter. Therefore, this law is absolutely correct only in limiting case of infinitesimally thin tube. It is also obvious that for a capillary of finite radius, the law gives a somewhat underestimated value for the height of liquid rise.

III. MENISCUS SHAPE IN A CAPILLARY TUBE: THE RIGOROUS TREATMENT

Refer to the cylindrical coordinate system for the meniscus profile as shown in figure 2.

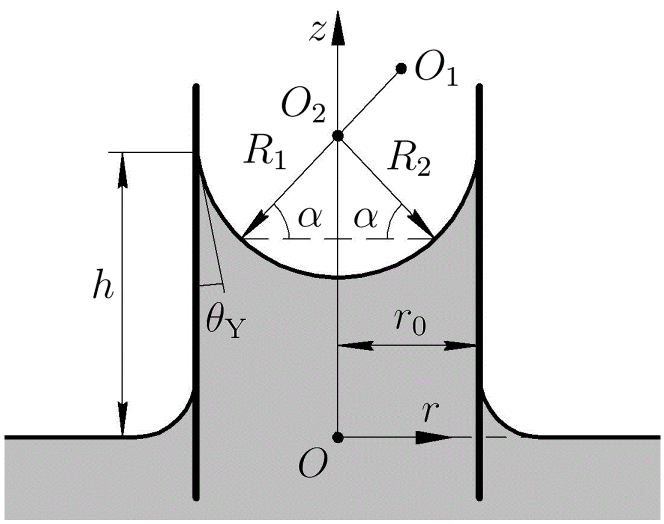

Figure 2. Schematic of meniscus profile in the axisymmetric tube.

We choose the origin O at the liquid level away from the tube. Assume the atmospheric pressure to be p_0. Then the pressure inside the liquid for the liquid-vapour points located at height z is the atmospheric pressure reduced by the hydrostatic pressure $\rho g z$ and increased by the Laplace pressure p_L, that is $p_0 - \rho g z + p_\mathrm{L}$. Outside the liquid, this pressure is simply equal to p_0. Under the equilibrium condition, the pressure inside and outside the liquid should be the same. It means that $p_\mathrm{L} = \rho g z$ (the Young-Laplace equation).

The general expression for the Laplace pressure [3] reads as

$$p_\mathrm{L} = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 2\gamma H, \quad (2)$$

where R_1 and R_2 are the principal radii of curvature (figure 2); H is the mean curvature of the surface. The value R_1 of the radius of curvature of a plane curve, which is given by the equation $z(r)$, can be found by the well-known equation [4]:

$$R_1(r) = \frac{\left[1 + (z'(r))^2\right]^{3/2}}{z''(r)}, \quad (3)$$

where $z'(r)$ and $z''(r)$ are, respectively, the first and second derivatives of function $z(r)$. Considering figure 2, we find

$$R_2(r) = \frac{r}{\cos\alpha} = \frac{r\sqrt{1 + (z'(r))^2}}{z'(r)}, \quad (4)$$

where $z'(r) = \tan\alpha$. Combining equations (2) - (4), we finally get

$$\frac{1}{\tilde{r}}\left(\frac{\tilde{r}\tilde{z}'}{\sqrt{1 + (\tilde{z}')^2}}\right)' = \tilde{r}_0^2\tilde{z}, \quad (5)$$

where $\tilde{z} = z/r_0$ and $\tilde{r} = r/r_0$ $(0 < \tilde{r} < 1)$ are the reduced (dimensionless) cylindrical coordinates; $\tilde{r}_0 = r_0/l_\mathrm{c}$ is the relative (dimensionless) capillary radius; $l_\mathrm{c} = \sqrt{\gamma/(\rho g)}$ is the capillary length. If $\tilde{r}_0 \ll 1$, the capillary forces dominate (thin tube). If $\tilde{r}_0 \gg 1$, the gravity dominates (wide tube). The capillary lengths less than 1 mm are rare, as they require extreme densities or very low surface tensions. For most liquids, l_c is in the range of 1–5 mm (for example, for water $l_\mathrm{c} \approx 3.83$ mm). The even more elegant derivation of equation (5), based on the minimization of the Helmholtz energy of the system, is presented in Ref. [5].

In equation (5) we represent the sum of two terms as derivative of a single function. The solution of equation (5) should satisfy the following two boundary conditions:

$$\left.\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{r}}\right|_{\tilde{r}=0} = 0, \quad (6)$$

$$\left.\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{r}}\right|_{\tilde{r}=1} = \cot\theta_\mathrm{Y}. \quad (7)$$

The advantage of equation (5) is that it is written in dimensionless form. This allows us to study the behavior of solutions depending on only two parameters of the theory: $\tilde{r}_0$ and θ_Y. Unfortunately, the non-linear second-order differential equation (5) in combination with boundary conditions (6) and (7) has no analytical solution. Therefore, a numerical method, i.e. the shooting method, should be adopted. This procedure can be realized using some advanced mathematical software (NumPy [6], SciPy [7], etc).

Equations (5)-(7) are invariant under the transformation $\tilde{z} \to -\tilde{z}$, $\theta_Y \to \pi - \theta_Y$, indicating the symmetry between a rising ($0 < \theta_Y < \pi/2$) and a depressing ($\pi/2 < \theta_Y < \pi$) meniscus [8]. In Ref. 9 it is strictly proved that for raising (depressing) meniscus the function $z(r)$ always increases (decreases) monotonically and does not contain inflection points. It means that the raising (lowering) meniscus is everywhere concave upward (downward).

IV. ASYMPTOTIC SOLUTIONS IN THE TWO LIMITING CASES

The asymptotic solution of equations (5)-(7) for $\tilde{r}_0 \ll 1$ can be obtained if we recall that in this case $z(r) \approx h_J$ ($\tilde{z}(\tilde{r}) \approx 2\cos\theta_Y/\tilde{r}_0^2$) and $\tilde{r}_0^2\tilde{z} \approx 2\cos\theta_Y = \text{const}$. Then equation (5) takes the following form:

$$\left(\frac{\tilde{r}\tilde{z}'}{\sqrt{1+(\tilde{z}')^2}}\right)' = 2\tilde{r}\cos\theta_Y. \tag{8}$$

Equation (8) can be transformed to the separable first-order differential equation satisfying both conditions (6) and (7):

$$\tilde{z}' = \frac{\cos\theta_Y \tilde{r}}{\sqrt{1-\cos^2\theta_Y \tilde{r}^2}}. \tag{9}$$

Its solution, that satisfies approximate condition $\tilde{z}(0) \approx 2\cos\theta_Y/\tilde{r}_0^2$, reads as

$$\tilde{z}(\tilde{r}) \approx \frac{2\cos\theta_Y}{\tilde{r}_0^2} + \frac{1}{|\cos\theta_Y|} - \sqrt{\frac{1}{\cos^2\theta_Y} - \tilde{r}^2}. \tag{10}$$

Therefore, for $\tilde{r}_0 \ll 1$ meniscus is the spherical cap with radius $R = r_0/|\cos\theta_Y|$. If $|\cos\theta_Y| \ll 1$, equation (10) simplifies to

$$\tilde{z}(\tilde{r}) \approx \frac{2\cos\theta_Y}{\tilde{r}_0^2} + \frac{\cos\theta_Y}{2}\tilde{r}^2. \tag{11}$$

Thus, for $\tilde{r}_0 \ll 1$ and $|\cos\theta_Y| \ll 1$ it is equally valid to regard the meniscus shape as either parabolic or spherical.

From a purely physical perspective, when $\tilde{r}_0 \gg 1$, it can be assumed that both the function $\tilde{z}(\tilde{r})$ and its derivative remain small in a central core region spanning most of the cylinder, except for a boundary-layer region near the wall. This enables the simplification of equation (6) to

$$\tilde{z}'' + \frac{\tilde{z}'}{\tilde{r}} \approx \tilde{r}_0^2\tilde{z}, \tag{12}$$

This equation can be easily reduced to the modified Bessel differential equation, whose solution satisfying the boundary condition (6) has the following form: $\tilde{z}(\tilde{r}) = C_2 I_0(\tilde{r}_0\tilde{r})$, where $I_0(x)$ is the modified zero-order Bessel function of the first kind. Using condition (7) and relation $I_0'(x) = I_1(x)$ ($I_1(x)$ is the modified first-order Bessel function of the first kind) we get $C_2 = \cot\theta_Y/(\tilde{r}_0 I_1(\tilde{r}_0))$ and

$$\tilde{z}(\tilde{r}) \approx \frac{\cot\theta_Y I_0(\tilde{r}_0\tilde{r})}{\tilde{r}_0 I_1(\tilde{r}_0)}. \tag{13}$$

V. THE MEAN CURVATURE

Taking into account the Young-Laplace equation and equation (2), we find the expression for the relative mean curvature:

$$\widetilde{H}(\tilde{r}) = \frac{H(\tilde{r})r_0}{\cos\theta_Y} = \frac{\tilde{r}_0^2}{2\cos\theta_Y}\tilde{z}(\tilde{r}). \tag{14}$$

In figure 3 we plot the value of $\widetilde{H}$ as a function of the radial distance $\tilde{r}$, calculated at different relative capillary radius $\tilde{r}_0$ and Young's contact angle θ_Y. In all possible cases, $\widetilde{H}(\tilde{r})$ is a monotonically increasing function, and $\widetilde{H}(1) > 1$ always holds. It is notable that across a wide range of $\tilde{r}_0$ and θ_Y values, the function $\widetilde{H}(\tilde{r})$ typically reaches a value of 1 at approximately $\tilde{r} \approx 0.7$.

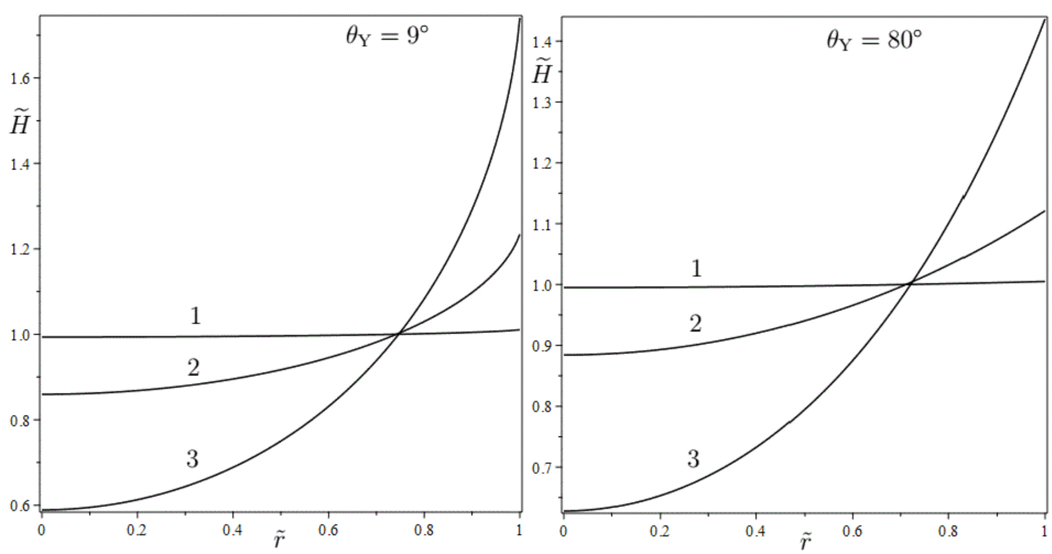

Figure 3. Dependence $\widetilde{H}(\tilde{r})$ at: (1) - $\tilde{r}_0 = 0.2$; (2) - $\tilde{r}_0 = 1$; (3) - $\tilde{r}_0 = 2$.

VI. THE COLUMN HEIGHT: APPROXIMATE VERSUS EXACT SOLUTIONS

The true column height h can be defined as the product $r_0\tilde{z}(1)$. Using equation (1), we find the column height within Jurin's approximation: $h_J = 2r_0 \cos\theta_Y/\tilde{r}_0^2$. Then, the relative error ε_h describing the deviations of Jurin's model from reality reads as:

$$\varepsilon_h = \frac{h - h_J}{h} = \frac{\tilde{z}(1) - 2\cos\theta_Y/\tilde{r}_0^2}{\tilde{z}(1)}. \tag{15}$$

In figure 4 we plot the value of ε_h as a function of the relative capillary radius, calculated at three different angles θ_Y.

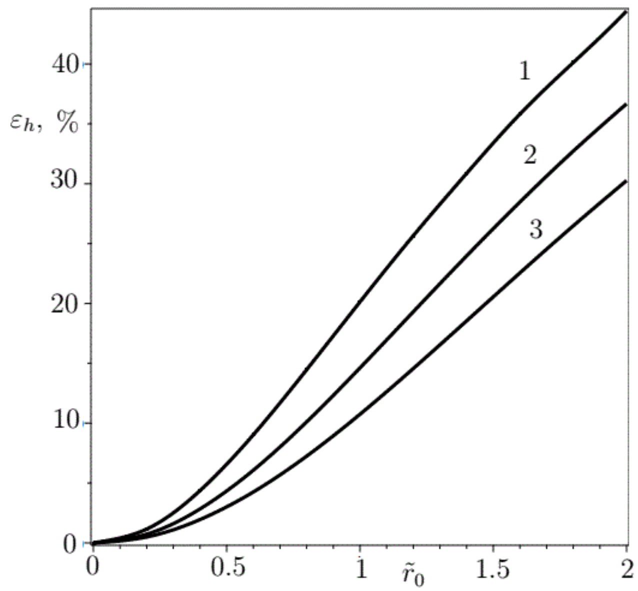

Figure 4. Dependence $\varepsilon_h(\tilde{r}_0)$ at: (1) - $\theta_Y = 5°$; (2) - $\theta_Y = 30°$; (3) - $\theta_Y = 85°$.

In all practical cases, there is a significant error already at $r_0 = l_c$. The deviations increase as Young's contact angle decreases.

The asymptotic analysis [8] gives us the true value of the liquid rise in the limiting case of very large $\tilde{r}_0$:

$$h_\infty \approx l_c \sqrt{2(1 - \sin\theta_Y)}. \tag{16}$$

Therefore, at $\tilde{r}_0 \to \infty$ the column height does not depend on the radius of the capillary at all and takes a constant finite value. For example, for water inside a glass tube $h_\infty \approx$ 5.41 mm. This feature is explained by the fact that at very large radii the meniscus is flat in the central region, and the rise of the liquid is determined exclusively by the edge effect at the wall, the area of action of which has a characteristic size l_c.

VII. CONCLUSIONS

In conclusion, we made a revision of the thin capillary model by means of an extensive investigation of the meniscus shape and column height. This paper not only refines the theoretical understanding of capillary action but also serves as a pedagogical tool to deepen students' appreciation of the interplay between physical principles and mathematical modelling. Finally, this topic has a research pattern (including numerical calculations and analysis) and can be successfully used in undergraduate courses or projects.

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