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ON THE VARIABLE MASS PROJECTILE MOTION

SOBRE EL MOVIMIENTO DE UN PROYECTIL CON MASA VARIABLE.

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We consider the features of the translatory motion of a rocket. It is found that its trajectory can have an inflection point or even contain an S-shaped section in the case of the relatively low initial speed. Both these features are explained by the fact that initially, the vertical component of the constant thrust is less than the gravity force, however, due to fuel consumption, it becomes greater than gravity. The times corresponding to the trajectory local extrema are determined using the Lambert W-function.

Consideramos las características del movimiento de traslación de un cohete. Se ha comprobado que su trayectoria puede tener un punto de inflexión o incluso contener una sección en forma de S en el caso de una velocidad inicial relativamente baja. Ambas características se explican por el hecho de que inicialmente la componente vertical del empuje constante es menor que la fuerza de gravedad, sin embargo, debido al consumo de combustible, se vuelve mayor que la fuerza de gravedad. Los tiempos correspondientes a los extremos locales de la trayectoria se determinan utilizando la función W de Lambert.

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I. INTRODUCTION

Projectile motion is one of the basic topics that are considered in the introductory mechanics course. There are many variations of this problem namely: drag-free projectile motion [1]; projectile motion in the presence of both linear [2] and quadratic drag [3], [4]; wind-influenced projectile motion [5]; projectile motion affected by the Magnus force [6]; relativistic projectile motion [7], the added mass of a spherical projectile [8], etc.

There are also many sources considering the examples of variable mass motion in the presence of gravity. Mungan and Lipscombe [9] analyze the interesting case of the vertical launch of a grappling hook. The features of the vertical launch of a rocket in a gravitational field are described in [10] and [11] (p. 139).

In this paper, we consider two-dimensional motion (projectile motion) with constant thrust and the simplest possible approximation in which the rocket mass linearly decreases with time. The problem is best suited for the beginning of an upper-level undergraduate course in classical mechanics.

II. THEORY

If we consider a rocket launched at an angle to the horizon with a constant thrust force throughout its ascent and descent phases, it is possible to achieve a translatory motion, where the reactive force remains in the same direction all the time. However, it's important to note that this scenario would require specific conditions and considerations.

To achieve a translatory motion, the rocket's thrust vector must be angled appropriately relative to its velocity vector at each point in its trajectory. By controlling the direction of the thrust, it is possible to balance the gravitational force and achieve a curved path that maintains a constant direction of the reactive force.

One way to achieve this is by employing a guidance system that adjusts the direction of the rocket's engines to maintain the desired trajectory. This guidance system continuously calculates the necessary adjustments to the thrust vector based on factors such as the rocket's position, velocity, and desired path. By dynamically changing the direction of the thrust, the rocket can follow a curvilinear path while keeping the reactive force consistently oriented in the same direction.

It is worth noting that achieving and maintaining such a trajectory requires sophisticated control systems, precise calculations, and real-time adjustments. Additionally, other factors like atmospheric conditions, external forces, and the rocket's aerodynamic properties influence the actual trajectory and make it challenging to achieve a perfect curvilinear translational motion. Another practically important variant of rocket motion is the case when the speed and thrust are always co-directed. However, this movement model is much more complex in terms of mathematical description.

Let us consider a body with the initial mass m_0 that moves in xy-plane under two forces: the jet force and the gravity force. Now, we should employ the equation of variable-mass motion (the rocket thrust equation). The reader may not be familiar with this equation. In this regard, we can suggest for the familiarization Ref. [11] (pp. 136-138) or Ref. [12], where a detailed derivation and analysis of this equation are presented (there is a small mistake in one slide of Ref. [12], that is corrected later in the video). In our case, the equation of variable-mass motion reads as:

$$m(t)\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{v}_{\mathrm{rel}} + m(t)\boldsymbol{g}, \tag{1}$$

where $\boldsymbol{v}$ is the rocket velocity; $\boldsymbol{v}_{\mathrm{rel}}$ is the exit velocity of the combustion products relative to the rocket. The first term on the right side of equation (1) is the thrust, while the second one represents the gravity force.

Assuming for simplicity that the body mass decreases linearly with time (this assumption gives us a constant thrust), we have:

$$m(t) = m_0 - kt \geq 0, \tag{2}$$

so that derivative $\mathrm{d}m/\mathrm{d}t = -k = \text{const}$ ($k > 0$). Since, $m(t)$ should remain positive all the the time, we limit the flight time to the interval $0 \leq t < m_0/k$.

We denote the constant angle between the thrust (that is, the rocket's symmetry axis) and x-axis as α_0 ($0 \leq \alpha_0 \leq \pi/2$). This means that the rocket performs a purely translatory motion. Although this case is not of great practical importance, the problem is easily solvable in such a formulation. In all other cases we deal with a set of non-linear differential equations of motion. In addition, in reality there will be a moment of air resistance force that will tend to rotate the rocket. This effect is caused by the fact that the center of mass and the center of air resistance do not coincide. To eliminate this effect, it will be necessary to use the system of active stabilization of the rocket flight.

The vector equation (1) splits into two independent scalar equations:

$$m(t)\frac{\mathrm{d}v_x}{\mathrm{d}t} = kv_{\mathrm{rel}}\cos\alpha_0, \tag{3}$$

$$m(t)\frac{\mathrm{d}v_y}{\mathrm{d}t} = kv_{\mathrm{rel}}\sin\alpha_0 - m(t)g. \tag{4}$$

Solving equations (2)-(4) along with initial condition $v(0) = v_0$, we get:

$$v_x = \left[v_0 - v_{\mathrm{rel}}\ln\left(1 - \frac{kt}{m_0}\right)\right]\cos\alpha_0, \tag{5}$$

$$v_y = \left[v_0 - v_{\mathrm{rel}}\ln\left(1 - \frac{kt}{m_0}\right)\right]\sin\alpha_0 - gt. \tag{6}$$

As far as the initial rocket velocity is non-zero, the model system used in this paper could be related with the movement of the second or further stage in a multi-stage launch. Using initial conditions $x(0) = 0$, $y(0) = 0$, and integrating equations (5) and (6) over time, we find:

$$x = \left[(v_0 + v_{\mathrm{rel}})t + v_{\mathrm{rel}}\left(\frac{m_0}{k} - t\right)\ln\left(1 - \frac{kt}{m_0}\right)\right]\cos\alpha_0, \tag{7}$$

$$y = \left[(v_0 + v_{\mathrm{rel}})t + v_{\mathrm{rel}}\left(\frac{m_0}{k} - t\right)\ln\left(1 - \frac{kt}{m_0}\right)\right]\sin\alpha_0 - \frac{gt^2}{2}. \tag{8}$$

At $v_{\mathrm{rel}} \to 0$ and $k \to 0$ (that is, in the case of zero thrust), expressions (5)-(8) are transformed into ordinary equations describing free projectile motion.

Let us introduce a set of dimensionless variables to reduce the number of input parameters to the problem and make the analysis of the equations of rocket motion simpler. We denote: $u_{x,y} = v_{x,y}/v_{\mathrm{rel}}$ is the dimensionless instant speed; $u_0 = v_0/v_{\mathrm{rel}} > 0$ is the dimensionless initial speed; $\tau = kt/m_0$ is the dimensionless time ($0 \leq \tau < 1$); $j = m_0 g/(kv_{\mathrm{rel}})$ is the dimensionless acceleration; $\rho_x = kx/(m_0 v_{\mathrm{rel}})$, $\rho_y = ky/(m_0 v_{\mathrm{rel}})$ are the dimensionless Cartesian coordinates. In this case, equations (5)-(8) can be rewritten as:

$$u_x = [u_0 - \ln(1-\tau)]\cos\alpha_0, \tag{9}$$

$$u_y = [u_0 - \ln(1-\tau)]\sin\alpha_0 - j\tau, \tag{10}$$

$$\rho_x = [(u_0 + 1)\tau + (1-\tau)\ln(1-\tau)]\cos\alpha_0, \tag{11}$$

$$\rho_y = [(u_0 + 1)\tau + (1-\tau)\ln(1-\tau)]\sin\alpha_0 - \frac{j\tau^2}{2}. \tag{12}$$

Considering equations (11), (12), and taking into account the equality

$$\lim_{\tau\to 1}(1-\tau)\ln(1-\tau) = 0, \tag{13}$$

we have for values of τ close to 1:

$$\rho_y \approx \rho_x \tan\alpha_0 - \frac{j}{2}. \tag{14}$$

Therefore, the idealized body's trajectory is enclosed between line $\rho_y = \rho_x \tan\alpha_0$ and the line given by equation (14) (figure 1).

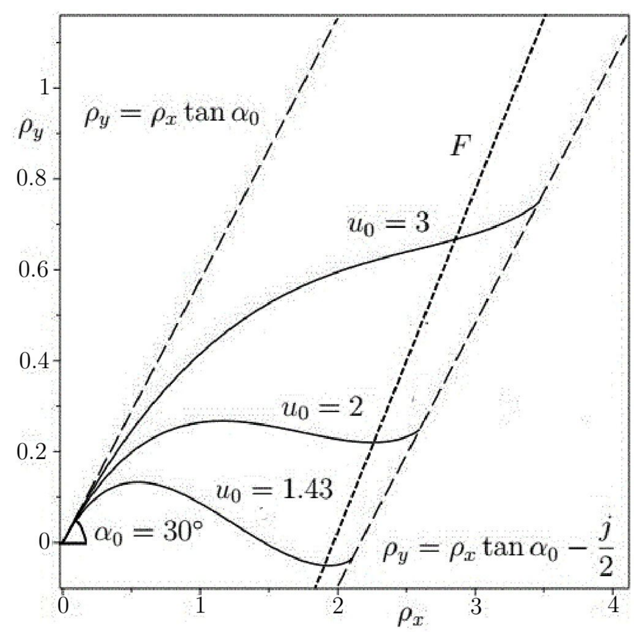

Figure 1. The body's trajectory at $\alpha_0 = 30°$, $j = 2.5$, and different values of u_0. The solid lines are drawn for $0 \leq \tau \leq 1$.

It should be noted that in reality the assumption $\tau \to 1$ is unattainable, since the rocket engine has non-vanishing mass.

For example, for rockets using dense components such as liquid oxygen and kerosene as fuel, the ratio of fuel mass to structural mass reaches 20 : 1. For rockets powered by oxygen and hydrogen, this ratio is about 10 : 1. Taking this fact into account, we introduce dashed line F in figure 1 to approximately separate sections of trajectories with non-zero fuel mass from those parts where our model no longer correctly describes the movement.

At fixed values of α_0, j, and relatively high value of u_0, curve $\rho_y(\rho_x)$ has an inflection point (figure 1). As u_0 decreases, two local extrema (maximum and minimum) appear on this curve. With a further decrease in the initial speed u_0, the minimum shifts to the region of negative values of ρ_y (figure 1).

The time points at which these local extrema are reached can be found by solving equation $u_y(\tau) = 0$, where function $u_y(\tau)$ is given by equation (10). We have:

$$\beta\tau = u_0 - \ln(1-\tau), \tag{15}$$

where $\beta = j/\sin\alpha_0 > 0$. The equation (15) can be rewritten as:

$$-\beta(1-\tau) = u_0 - \beta - \ln(1-\tau). \tag{16}$$

Converting this equation to exponential form, we derive:

$$-\beta(1-\tau)\exp(-\beta(1-\tau)) = -\beta\exp(u_0-\beta). \tag{17}$$

Expression (17) is the equation relative to unknown variable τ and it has the formal form: $w\exp(w) = z$, where $w = -\beta(1-\tau)$, $z = -\beta\exp(u_0-\beta)$. The solution to this transcendental equation (relative to variable w) is not expressed in terms of elementary functions and is known as the multivalued Lambert $W(z)$-function (also called the omega function, product logarithm, or the inverse function) [13], [14] with respect to the variable z. Thus,

$$-\beta(1-\tau) = W\left[-\beta\exp(u_0-\beta)\right]. \tag{18}$$

Then

$$\tau = \frac{W\left[-\beta\exp(u_0-\beta)\right]+\beta}{\beta}. \tag{19}$$

Since, in our case both argument z and solutions $W(z)$ should be real numbers (moreover, z is always negative), we deal with only two branches of $W(z)$ [14]. The principal branch (W_0) corresponds to the local minimum and gives the values between -1 and 0, whereas the lower branch (W_{-1}) corresponds to the local maximum and gives the values below -1. Therefore,

$$\tau_{\max} = \frac{W_{-1}\left[-\beta\exp(u_0-\beta)\right]+\beta}{\beta}, \tag{20}$$

$$\tau_{\min} = \frac{W_0\left[-\beta\exp(u_0-\beta)\right]+\beta}{\beta}, \tag{21}$$

In order to put in context the results, we turn back to the original variables in equations (20), (21):

$$t_{\max} = \frac{m_0}{k} + \frac{v_{\mathrm{rel}}\sin\alpha_0}{g} W_{-1}\left[-\frac{m_0 g}{k v_{\mathrm{rel}}\sin\alpha_0}\exp\left(\frac{v_0}{v_{\mathrm{rel}}} - \frac{m_0 g}{k v_{\mathrm{rel}}\sin\alpha_0}\right)\right], \tag{22}$$

$$t_{\min} = \frac{m_0}{k} + \frac{v_{\mathrm{rel}}\sin\alpha_0}{g} W_0\left[-\frac{m_0 g}{k v_{\mathrm{rel}}\sin\alpha_0}\exp\left(\frac{v_0}{v_{\mathrm{rel}}} - \frac{m_0 g}{k v_{\mathrm{rel}}\sin\alpha_0}\right)\right]. \tag{23}$$

It is known [14] that for negative values of z these two branches of $W(z)$-function are defined only on interval $-1/\exp(1) \leq z \leq 0$. Then, the condition for the existence of both extrema is the following inequality: $\beta > \exp(\beta - u_0 - 1)$ or

$$u_0 < \beta - \ln\beta - 1. \tag{24}$$

For $u_0 = \beta - \ln\beta - 1$, $\tau_{\max} = \tau_{\min} = (\beta-1)/\beta$ (here we used the special values of two branches: $W_0(-1/e) = W_{-1}(-1/e) = -1$). If $u_0 > \beta - \ln\beta - 1$, both extrema are absent. The above analysis implies that $\beta > 1$. For $0 < \beta < 1$ both solutions (20) and (21) are negative and are not relevant for our consideration.

Figure 2 shows the dependences of the solutions of equations (20) and (21) (the extrema points τ_{extr}) on u_0 constructed at two different values of β. At fixed value of β, $\tau_{\max}$ ($\tau_{\min}$) is a monotonically increasing (decreasing) function of u_0. As β increases, the values of $\tau_{\max}$ ($\tau_{\min}$) decrease (increase) rapidly.

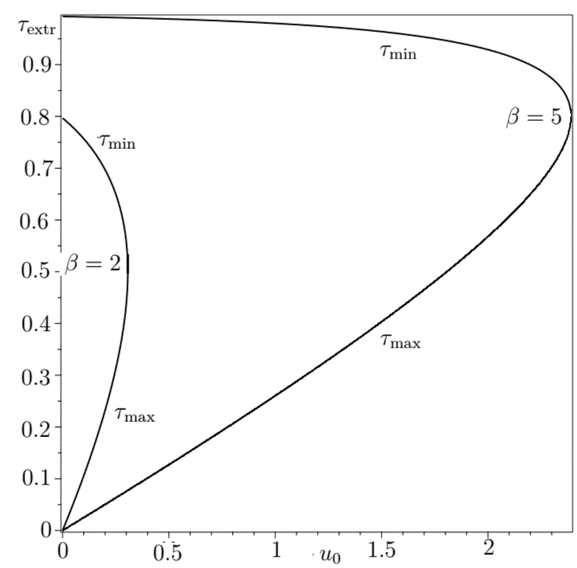

Figure 2. The solutions (20), (21) as functions of u_0 at two different values of β.

III. DISCUSSION

Let us consider equations (3) and (4). Equation (4) tells us that under the influence of the horizontal component of the thrust v_x increases all the time. At the same time, the vertical component v_y of the velocity can change non-monotonically. Indeed, if the vertical component of the thrust ($k v_{\mathrm{rel}}\sin\alpha_0$) is greater all the time than the gravity force ($m_0 g$), then v_y increases all the time. If $k v_{\mathrm{rel}}\sin\alpha_0 < m_0 g$, then v_y is diminishing from the start. However, due to fuel consumption, the mass of the rocket decreases and at a certain point in time t_0, v_y begins to increase. This time can be found from the condition $kv_{\rm rel}\sin\alpha_0 = (m_0 - kt_0)g$:

$$t_0 = \frac{m_0}{k} - \frac{v_{\rm rel}\sin\alpha_0}{g}, \tag{25}$$

or $\tau_0 = 1 - 1/\beta$. The time t_0 corresponds to the inflection point on the trajectory (figure 1).

If, moreover, $v_0\sin\alpha_0$ is relatively small, then v_y has time to go to zero (a local maximum on the trajectory) at time t_0 and then become negative (the rocket will descend for some time). Subsequently, v_y becomes positive again and the rocket begins to ascend (the beginning of this interval corresponds to a local minimum on the trajectory).

We can provide a physical motivation for equation (22) by asking what the minimum launch velocity u_0 is such that the rocket will not fall back down and so the curves in figure 1 show no extrema. It follows from figure 2 that for $\beta = 5$ the minimum value of u_0 is equal to 2.5, whereas for $\beta = 2$ the minimum value of u_0 is approximately equal to 0.35.

It is seen from figure 1 that for $u_0 = 1.43$, ρ_y turns negative meaning that the rocket crashes unless you have launched it off a cliff. An interested student might ask "What is the minimum initial velocity to avoid $\rho_y < 0$?". Similarly, a student might wonder what the trajectory looks like when dimensionless time is large enough that the rocket runs out of fuel. We can suggest these topics as students' homework or even as an independent project.

IV. CONCLUSIONS

The variable mass projectile motion is a topic that is possibly not explored deeply enough in undergraduate physics degree curricula. In this paper, we try to fill this gap. The topic may help students to better grasp such important physical concepts as translational motion, thrust, trajectory, projectile motion, etc. Finally, our consideration should help readers to observe the differences between the classical projectile motion and the trajectory of a rocket.

The topic can be recommended to students for further reading when studying dynamics of systems with varying mass or in physics electives. This issue can also be used as an undergraduate project.

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