Rev. Cub. Física vol. 24 No. 2 (2007) p.144-149
ISSN: 0253-9268. Original paper

Revista Cubana de Física
Calle I No. 302 e/ 15 y 17
Vedado, La Habana.
www.fisica.uh.cu/biblioteca/revcubfi/index.htm

Mechanical interpretation of existence theorems in a nonlinear Dirichlet problem

Augusto González

Instituto de Cibernética, Matemática y Física, Calle E 309, Vedado, Ciudad de La Habana, Cuba;
agonzale@icmf.inf.cu

Recibido el 23/06/200. Aprobado en versión final el 2/12/07

Sumario. La existencia de soluciones radiales a un problema de Dirichlet no lineal en una región esférica es traducida al lenguaje de la Mecánica, es decir se expresa como requerimientos sobre el tiempo en que se mueve una partícula en un potencial externo y sujeta a la acción de una fuerza viscosa. Esta forma de abordar el problema nos brinda un método cualitativo, pero riguroso, de analizar el caso general. Teoremas conocidos son fácilmente reproducidos. Se dan ejemplos de nuevos teoremas, los cuales prueban la utilidad de este método cualitativo.

Abstract. The existence of radial solutions to a nonlinear Dirichlet problem in a ball is translated to the language of Mechanics, i.e. to requirements on the time of motion of a particle in an external potential and under the action of a viscosity force. This approach provides a qualitative, but rigorous, method for the analysis of the general case. Known theorems are easily reproduced and examples of new theorems are given, which prove the usefulness of this qualitative method.

Palabras claves: Ordinary differential equations 02.30.Hq, Nonlinear dynamics and chaos 05.45.-a, Formalisms in classical mechanics 45.20.-d.

1 Introduction

In the present paper, we consider the following nonlinear Dirichlet problem:

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2}$$

where f is a differentiable function and Ω is the ball of radius R in R^D. We look for conditions guaranteeing the existence of spherically symmetric solutions to (1-2).

The above mentioned problem has been extensively studied in the past (see, for example, ref. [1-5] and references therein). In this paper, our purpose is to develop a very simple picture, based on Mechanics, for the analysis of the existence of solutions to (1-2). This qualitative picture reproduces the existing results and, in principle, provides a frame for the analysis of the radial solutions to (1-2) in the presence of an arbitrary nonlinear function f. Examples of new theorems are given, which show the usefulness of the method.

To our knowledge, the analogy of the radial equation (1) with the Newtonian law of motion of a particle was first used by Coleman[6] to obtain the approximate form of the solution connecting false and true vacua in scalar field theories. This solution enters the semiclassical expression for the decay probability of the false vacuum state. Application of this analogy to the analysis of the existence of solitary waves in nonlinear one-dimensional media has proven to be very useful too[7].

The plan of the paper is as follows. In the next Section, the problem about the existence of solutions to (1-2) is translated to the language of Mechanics. Two limiting solvable cases, the one-dimensional problem and the linear equation, are considered and a few general results are given. Let us stress that the function $f(u)$ is interpreted as the derivative of a potential, thus the linear equation describes the motion in a quadratic potential. Section 3 deals with potentials having a well around $u = 0$. The most interesting examples studied in this Section are, in our opinion, the potentials with barriers. In Section 4, we study the motion in a potential with a hill around $u = 0$. In Section 5, we consider singular (finite and infinite) potentials. Concluding remarks are given at the end of the paper.

2 The analogy with mechanics

We start by considering the spherically symmetric version of Problem (1-2):

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{D-1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + f(u) = 0, \tag{3}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r}(0) = 0, \qquad u(R) = 0. \tag{4}$$

Written in this form, the analogy with Mechanics is evident. Equation 3 is nothing, but the Newton law for a particle of unit mass moving in a potential V which is the antiderivative of f, $f(u) = dV/du$, and under the action of a viscosity force inversely proportional to time. The particle should start with zero velocity from a position $u(0)$ and arrive to $u = 0$ in a time R (Fig. 1(*a*)).

We have drawn in Fig. 1(*b*) a generic positive solution to (3-4) for a given V. In general, the particle will realize damped oscillations around the point $u = 0$ (Fig. 2). Let T_n $(u(0))$ be the time the particle spends to reach the point $u = 0$ n times starting from $u(0)$. Thus, the existence of a solution to (3-4) may be formulated in the following terms:

"In the potential V, there exists a $u(0)$ and a positive integer, n, such that $T_n(u(0)) = R$"

The interesting point is that in many cases we may perform simple estimates, based on physical principles, of the dependence T_n vs $u(0)$ and, consequently, we may give criteria for the existence of solutions to (3-4).

Let us first study two limiting cases in which equation (3) may be solved exactly. They will be very useful in the analysis below.

2.1 The one-dimensional (D = 1) problem. The $D = 1$ case is characterized by the absence of friction. Thus, the energy $E = (1/2)(du/dr)^2 + V(u)$ is conserved, $dE/dr = 0$, and the dependence $r(u)$ may be expressed in the form of an integral in each interval where du/dr does not change sign,

$$r - r_a = \int_{r_a}^{r} \mathrm{d}t = \{\mathrm{sign}(\mathrm{d}u/\mathrm{d}r)\} \int_{u_a}^{u} \frac{\mathrm{d}x}{\sqrt{2(V(u(0)) - V(x))}}. \tag{5}$$

In such conditions, the motion of a particle in a well is a periodic motion characterized by the function T_1

$$T_1(u^+(0)) = \int_0^{u^+(0)} \frac{\mathrm{d}x}{\sqrt{2(V(u^+(0)) - V(x))}}. \tag{6}$$

(For negative $u(0)$ the integration limits shall be reversed). Note that T_n may be expressed in terms of T_1:

$$T_n(u^+(0)) = \left(2[\frac{n+1}{2}] - 1\right)T_1(u^+(0)) + 2[\frac{n}{2}]T_1(u^-(0)), \tag{7}$$

where $[q]$ means the integer part of q, and $u^-(0)$ is defined from $V(u^+(0)) = V(u^-(0)) = E$.

For a given potential, the equation $T_n = R$ may be explicitly written and the existence of solutions to Problem (3 - 4) may be explicitly investigated.

We are not going to give further details of the analysis in this simple case and turn out to the higher dimensional ($D > 1$) problem, i.e. motion with friction. In this situation, there is another exactly solvable problem: the motion in a quadratic potential.

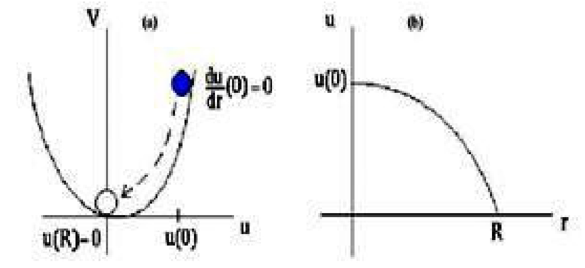

Figure 1. (*a*) The analogy with Mechanics, (*b*) A positive solution to (3 - 4) corresponding to the situation depicted in (*a*).

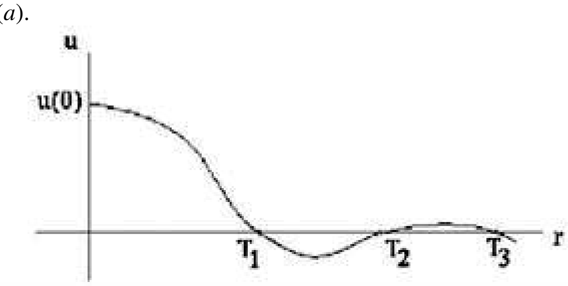

Figure 2: A generic damped oscillating function $u(r)$ describing the motion of a particle in V.

2.2 Motion in a quadratic potential (The linear equation). We consider a quadratic potential $V(u) = \frac{1}{2}\lambda u^2$. The equation of motion (3) takes the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} = -\lambda u - \frac{D-1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}. \tag{8}$$

The solution of this Eq. with initial condition $du/dr\,(0) = 0$ is expressed as

$$r^{1-n/2} J_{|D/2-1|}(\sqrt{\lambda} r)$$

where J is the Bessel function[8]. It is important to note that the main properties of the solution may be understood simply from the invariance properties of Eq. (8).

LEMMA: T_n does not depend on $u(0)$ and is proportional to $\lambda^{-1/2}$.

PROOF: The Eq. is invariant under a change in the scale of u, and also under the transformation $r \rightarrow C_r r$, $\lambda \rightarrow C_\lambda \lambda$, where $C_r = C_\lambda^{-1/2}$.

According to this Lemma, the function T_n $(u(0))$ takes a fixed value that depends only on λ and n. Varying appropriately the parameter λ (the potential), one may fulfill the requirement $T_n = R$. The corresponding set of parameters, $\{\lambda_n\}$, defines the eigenvalues of the linear problem.

2.3 Some useful results. In this subsection, we derive a few general results following from the analogy with Mechanics and classify the potentials to be studied.

In the presence of dissipation, the rate of change of the energy is written as

$$\mathrm{d}E/\mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}r}\Big((1/2)(\mathrm{d}u/\mathrm{d}r)^2 + V(u)\Big) = -\frac{D-1}{r}(\mathrm{d}u/\mathrm{d}r)^2 < 0, \quad (9)$$

i.e. $u(r)$ is damped, as mentioned above. It means that $E(u(0)) = V(u(0)) > E(0) > V(0) = 0$ (we have supposed that f is integrable, so that $V(u)$ may be defined as $\int_0^u f(x)\,\mathrm{d}x$). Then, we arrive at the following

THEOREM (A necessary condition): If $u(r)$ is a solution to (3 - 4) and f is integrable, then $u(0)$ is such that

$$\int_0^{u(0)} f(x)\,\mathrm{d}x > 0, \quad \mathrm{sign}(f(u(0))) = \mathrm{sign}(u(0)).$$

The last condition on the sign of $f(u(0))$ means that the particle shall be pushed towards the origin at the initial position, $u(0)$. Otherwise, it will never move to the origin passing through $u(0)$ because of the energy losses.

More sophisticated versions of this Theorem will be formulated below when studying potentials with barriers.

A second important result concerns the retardation effect of friction. Let us suppose that the particle moves from u_a to u_b. The time it spends in this motion may be written as

$$r_b - r_a = \int_{u_a}^{u_b} \frac{\mathrm{d}x}{\sqrt{(\mathrm{d}u_a/\mathrm{d}t_a)^2 + 2V(u_a) - 2V(x) - 2(D-1)\int_{r_a}^{t} \frac{\mathrm{d}\tau}{\tau}(\mathrm{d}x/\mathrm{d}\tau)^2}}. \quad (10)$$

Of course, this is not a closed expression because the derivative in the time interval (r_a, r_b) enters the r.h.s. of it. However, it is evident that

$$r_b - r_a > \int_{u_a}^{u_b} \frac{\mathrm{d}x}{\sqrt{(\mathrm{d}u_a/\mathrm{d}t_a)^2 + 2V(u_a) - 2V(x)}}, \quad (11)$$

i.e.

LEMMA: The time interval $r_b - r_a$ is greater than the time the particle spends to move from u_a to u_b without friction.

Finally, let us classify the potentials according to their properties in the neighborhood of $u = 0$. In the present paper, we will study four classes of potentials having different behaviors in the vicinity of this point (Fig. 3):

(a) The wells are defined as concave potentials around $u = 0$.

(b) The hills are convex around $u = 0$. Of course, at large $|u|$, $V(u)$ shall be positive (the necessary condition).

(c) and (d) correspond to singular potentials.

We will study below each class of potentials separately.

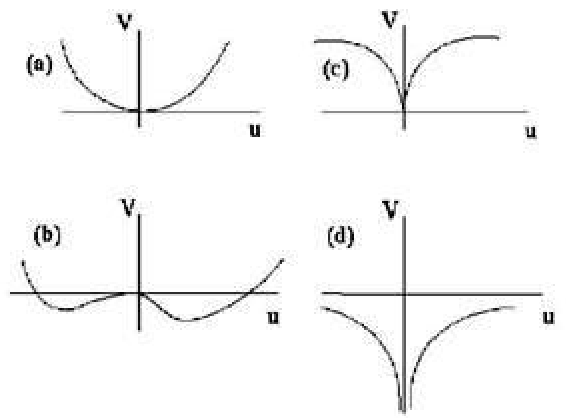

Figure 3. Different possibilities for the neighborhood of $u = 0$: (*a*) Well, (*b*) Hill, (*c*) Finite, but singular, (*d*) Infinite, singular potential.

3 Wells around $u = 0$

A well is defined as a region with only one local extremum, the minimum at $u = 0$. In this Section, we study some examples of potentials having a well around $u = 0$.

3.1 Potentials, quadratic in u = 0 and $|u| \to \infty$

Let $V(u)$ be a potential such that

$$V|_{u\to 0} \approx \frac{1}{2}\lambda(0)u^2,$$
$$V|_{|u|\to\infty} \approx \frac{1}{2}\lambda(\infty)u^2, \quad (12)$$

additionally, we will assume that the only zero of f is at $u = 0$. Then, we have the following

THEOREM: If $\lambda(0) < \lambda_1$ and $\lambda(\infty) > \lambda_k$, then Problem (3 - 4) has at least $2k + 1$ solutions.

This Theorem was obtained in ref. [5]. We will give a detailed proof of it by means of our method as an illustration. For the incoming Theorems, the proof will be shortened.

The statement is that the function T_n vs $u(0)$ has the form depicted in Fig. 4 for $1 \le n \le k$, i.e. for each T_n there are two solutions.

Indeed, the very small amplitude motion is governed by the $u \to 0$ behavior of V. T_n depends very smoothly on $u(0)$ in this region and $T_n \ge T_1 > R$. The latter inequality comes from $\lambda(0) < \lambda_1$.

On the other hand, the large amplitude motion is governed by the $|u| \to \infty$ asymptotic behavior and, according to the inequality $\lambda(\infty) > \lambda_k$, we have $T_n \ge T_k < R$. The point to clarify is why T_n for large $u(0)$ is not affected by the small-u behavior of V.

The answer is that, when $u(0)$ is large, the time the particle spends to move in the small-u region is negligible. This result comes from the scale invariance of the quadratic potential as shown in Fig. 5. Shadowed areas correspond to motion in the region $|u| < u_a$. It is seen that when $|u(0)| \to \infty$ the time spent in this motion shrinks to zero. It means that one can deform $V(u)$ at low $|u|$ without changing significantly T_n.

Thus, Problem (3 - 4) has 2k nontrivial solutions plus the trivial $u = 0$.

3.2 Potentials with barriers. In the previous Subsection, we assumed continuity of T_n vs $u(0)$. However, continuity is broken when V has local extrema, others than $u = 0$. The point is that, as may be seen from Eq. (5) and the retardation Lemma of Section 2.3, the time the particle spends to move out of a local maximum tends to infinity when $u(0)$ approaches the position of the maximum.

Then, let us first suppose that f has a unique second zero at a point $a > 0$. The following Theorem may be formulated

THEOREM. If $\lambda(0) > \lambda_k$ and $V|_{u(0) \to -\infty} > V(a)$, then Problem (3 - 4) has at least k solutions with $0 < u(0) < a$.

To prove it, we draw again the function $T_n(u(0))$, with $1 \le n \le$ k and positive $u(0)$. At low $u(0)$, $T_n \le T_k < R$. When $u(0) \to a$ from below, $T_n \to \infty$. The condition on $V(-\infty)$ guarantees that the motion is oscillatory around $u(0)$ and the particle does not escape to $-\infty$.

Note that it is difficult to draw the dependence T_n vs $u(0)$ for negative $u(0)$ without knowledge of the potential. The following Theorem, contained in ref. [5], states that for asymptotically quadratic potentials one can say much more.

THEOREM. If f has positive zeros, the first of which is at $u = a$, and $\lambda(0)$, $\lambda(\infty) > \lambda_k$, then Problem (3 - 4) has at least $4k - 1$ solutions.

We have drawn in Fig. 6 the potential and the functions T_1, T_n, $1 \le n \le$ k. The points b_+ and b_- are defined in the monotone regions. They satisfy $V(a) = V(b_+) = V(b_-)$. Dashed lines means that the curves are conditionally drawn, while shadowed intervals of $u(0)$ mean physically impossible initial conditions.

The dependence of T_1 on $u(0)$, when $0 < u(0) < a$, is the same as in the previous Theorem. For very large positive $u(0)$, T_1 is determined by $\lambda(\infty)$, i.e. $T_1|_{u(0) \to \infty} < T_k < R$. On the other hand, because of energy losses, if the particle starts from b_+ it will not reach the origin. By continuity, there exists $c_1 > b_+$ such that the particle arrives at a with zero velocity. This corresponds to an infinite T_1. When $u > c_1$ the particle reaches the origin and the dependence $T_1(u(0))$ is shown. Note that we can not say anything about T_1 for negative $u(0)$. Thus, the equation $T_1 = R$ will have, at least, two solutions.

Similar arguments are used in the analysis of T_n, $1 < n \le k$. C_n is now defined such that when $u(0) > c_n$ the origin is reached n times. Note that $T_n(c_n) = \infty$ and also that $c_1 = c_2 < c_3 = c_4 < c_5 \ldots$ On the l.h.s. of the origin, we can define the points $e_n < d < b_-$ d is such that when the particle arrives to a it does so with zero velocity, while e_n is such that for $u(0) < e_n$, the particle reaches the origin n times. Note that $e_2 = e_3 > e_4 = e_5 > e_{6\ldots}$ In other words, for each n there are 4 solutions. This proves the theorem.

Notice that, unlike papers in ref. [1-5], we are able to indicate forbidden regions for $u(0)$. This is a generalization of the necessary condition of Section 2.3.

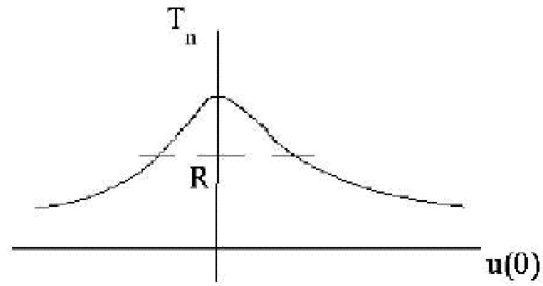

Figure 4. Dependence T_n vs $u(0)$ for the potential considered in Section 3.1.

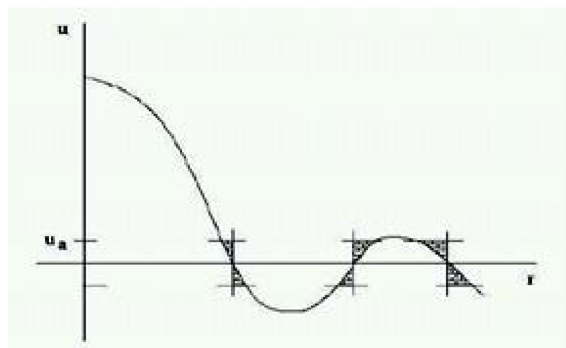

Figure 5. A consequence of the scale invariance of the quadratic potential. The shadowed areas correspond to motion in the region $|u| < u_a$. When $|u(0)| \to \infty$, the time spent in this motion shrinks to zero.

3.3 The potentials $V = g|u|^\beta$. Let us now consider the potentials $V = g|u|^\beta$, with $g > 0$, $\beta > 1$. We shall first prove that, whatever β be, $u(r)$ will have the form drawn in Fig. 2. After that, we will use scale-invariance properties of the equation of motion to obtain the dependence T_n vs $u(0)$. Let us prove the following general

LEMMA. In a potential well, $u(r)$ is an oscillating function of decaying amplitude.

PROOF. It is evident that the particle will reach the origin whatever the initial position is. It can not stop in an intermediate point where the force is not zero. Thus, the question is how long it takes to reach the origin and what is the final velocity. If this time and the velocity are finite, we can repeat the argument to conclude that $u(r)$ will have infinite zeros.

Let r_a be an intermediate time such that $|du/dr\,(r_a)| > 0$. Due to the particular form of the friction, we can obtain an upper bound for the time to reach the origin starting from $u(r_a)$, r_b, and a lower bound for $|du/dr(r_b)|$, if we neglect the potential for $r > r_a$ and solve the problem:

$$\frac{d^2u}{dr^2} + \frac{D-1}{r}\frac{du}{dr} = 0 \quad , \tag{13}$$

$$u(r_a) = u_a, \quad \frac{du}{dr}(r_a) = v_a \quad , \tag{14}$$

which has the following solution

$$\frac{\mathrm{d}u}{\mathrm{d}r}(r) = v_a(r_a/r)^{D-1} \quad , \tag{15}$$

$$u(r) = u_a + v_a\frac{r_a^{D-1}}{D-2}\left\{r_a^{2-D} - r^{2-D}\right\} \quad . \tag{16}$$

It means that $u = 0$ will be reached in a finite r_b with a finite velocity and $u(r)$ will have infinite zeros.

Thus, let us now turn out to the dependence T_n vs $u(0)$ in the potentials $V = g\,|u|^{\beta}$. The equation of motion takes the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} = -g\beta(\mathrm{sign}\ u)|u|^{\beta-1} - \frac{D-1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} \quad . \tag{17}$$

The properties of T_n following from the scale invariance of the equation are given in the next Lemma:

LEMMA. For fixed g, $T_n \sim |u(0)|^{1-\beta/2}$, while for fixed $u(0)$, $T_n \sim g^{-1/2}$.

Thus, for every n, the equation $T_n = R$ will have two solutions, and we arrive to the following

THEOREM. Problem (3-4) with $V = g\,|u|^{\beta}$, $g > 0$, $\beta > 1$ has infinite solutions[3].

One can now combine these with previous results. In quality of example, let us formulate the following

THEOREM. Let $\lambda(0) < \lambda_k$ and $V|_{|u|\to\infty} \sim |u|^{\beta}$, with $\beta > 2$, then solutions to Problem (3-4) with any $n \geq k$ zeros exist.

The curve T_n vs $u(0)$ for $n \geq k$ may be easily drawn in this case. Note that the dependence of $T_n|_{|u|\to\infty}$ on the low-$u(0)$ properties of V is, for $\beta > 2$, weaker than in the quadratic potential.

4 Hills around $u = 0$

We now study the motion in a potential like that one shown in Fig. 3 b. For simplicity, we assume that V is quadratic near zero ($\lambda(0) < 0$) and also quadratic at large values of u. No additional zeros of f exist. Then one can formulate the following

THEOREM. If $\lambda(0) < 0$ and $\lambda(\infty) > \lambda_k$, then Problem (3 - 4) has $2k+1$ solutions.

We have drawn in Fig. 7 the curve T_n vs $u(0)$ for $1 \leq n \leq k$. The large-$u(0)$ behavior of it is evident. The points b_+ and b_- are the zeros of V. The points c_n and e_n are defined as in the previous Section, i.e. starting from the right of c_n (the left of e_n) the particle may reach the origin n times.

Note that would it start from c_n (e_n), then it would arrive to $u = 0$ with zero velocity, i.e. $T_n(c_n) = T_n(e_n) = \infty$. Note also that $b_+ < c_1 < c_2 \ldots$, $b_- > e_1 > e_2 > \ldots$ Thus, for each n there are two solutions and the Theorem is proved.

Other potentials could be analyzed, but we think the given example is enough to show the advantages of the method.

5 Singular potentials

The main property of the singular potentials, Figs. 3 *c*) and *d*), is that the force, $-dV/du$, at $u = 0$ is ill-defined. So, the motion is not well defined right after the particle reaches the origin, and we can only analyze the existence of positive solutions to (3 - 4).

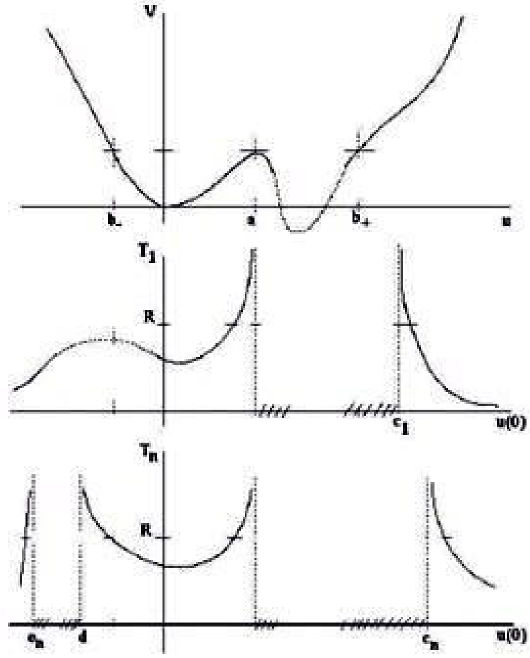

Figure 6. A potential with barriers and the corresponding $T_1(u(0))$, $T_n(u(0))$, $1 < n \leq k$.

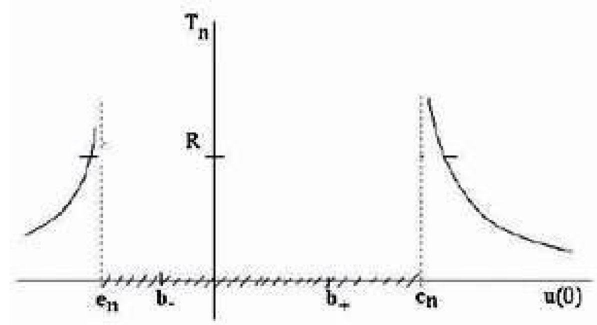

Figure 7. The curves T_n vs $u(0)$, $1 \leq n \leq k$, for the potential of Section 4. Notations are the same as in Fig. 6.

An example of a potential like 3 *c*) is $V = g\,|u|^{\beta}$, with $g > 0$ and $0 < \beta < 1$. Let us stress that the upper bound for r_b and the dependence $T_1 \sim |u(0)|^{1-\beta/2}$, obtained in the Lemmas of Section 3.3, are valid, so that the equation $T_1 = R$ has always a solution in this case.

The same analysis holds for the potential $V = -g\,|u|^{-\beta}$, with g, $\beta > 0$. This is a potential of the form 3 *d*). Scale invariance in this case leads to $T_1 \sim |u(0)|^{1+\beta/2}$, so that the equation $T_1 = R$ will always have a solution too.

We can now combine possibilities to obtain interesting situations. Let, for example, the potential V be quadratic at the origin with $\lambda(0) > 0$, while at long distances $V \sim V_0 - g\,|u|^{-\beta}$. No zeros of f exist, except the trivial at $u = 0$. Then, we obtain the following

THEOREM. If $\lambda(0) > \lambda_k$, then Problem (3 - 4) has at least $2k + 1$ solutions. The proof is trivial.

3 Concluding remarks

In the present paper, we used the analogy of Eq. (3) with the second Newton's law in order to obtain existence theorems to Problem (3 - 4). In addition to reproducing existing results, we give new examples of potentials (of *f*) in which it is relatively easy to analyze the existence of solutions.

We think the given examples show that the method is general enough to provide a first insight to the problem for any reasonable function *f*. After that, we may go further on in two ways: i) Use more rigorous methods to complete the proof and/or ii) Obtain numerical solutions to the equation.

Acknowledgments

The author acknowledges support by the Caribbean Network for Quantum Mechanics, Particles and Fields (ICTP-TWAS).

References

1. M. Esteban, Multiple Solutions of Semilinear Elliptic Problems in a Ball, J. Diff. Eqs. 57 112 - 137, (1985)
2. D.G. Costa and D.G. de Figueredo, Radial Solutions for a Dirichlet Problem in a Ball, J. Diff. Eqs. 60 80 - 89, (1985)
3. A. Castro and A. Kurepa, Infinitely Many Radially Symmetric Solutions to a Superlinear Dirichlet Problem in a Ball, Proc. Amer. Math. Soc. 101 57 - 64, (1987)
4. S. Kichenassamy and J. Smoller, On the existence of radial solutions of quasilinear elliptic equations, Nonlinearity 3 677 - 694, (1990)
5. A. Castro and J. Cossio, Multiple radial solutions for a semilinear Dirichlet problem in a ball, Revista Colombiana de Matematicas 27 15- 24, (1993)
6. S. Coleman, Fate of the false vacuum, Phys. Rev. D 15 2929 - 2934, (1977)
7. J. Gonzalez and J.A. Holyst, Solitary waves in one-dimensional damped systems, Phys. Rev. B 35, 3643 - 3646, (1987)
8. M. Abramowitz and I. Stegun, (Handbook of Mathematical Functions, Chapter 9, Dover Publications, New York, 1972).